ON NILPOTENT - INVARIANT MODULES

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Abstract: A module is called nilpotent-invariant if it is invariant under any nilpotent endomorphism of its injective envelope. The class of these modules is studied, and some of their properties are developed. In this paper, we continue to study nilpotent-invariant modules over general envelopes.

Keywords: Automorphism-invariant; endomorphism-invariant; nilpotent-invariant.

1 Introduction

Throughout this article, all rings are associative rings with identity and all modules are right unital. For a submodule N of M, we use $N \leq M$ ($N < M$) to mean that N is a submodule of M (respectively, proper submodule). We usually write $\mathrm{End}(M)$ ($\mathrm{Aut}(M)$) to indicate its ring of right $R-$module endomorphisms (respectively, automorphism). A submodule N of a module M is called small in M (denoted as $N \ll M$) if $N + K \neq M$ for any proper submodule K of M.

Let $\mathcal{X}$ be a class of right $R-$modules. We say that $\mathcal{X}$ is closed under isomorphisms, if $M \in \mathcal{X}$ and $N \cong M$, then $N \in \mathcal{X}$. Recall that, let $\mathcal{X}$ be a class of right R-modules which is closed under isomorphisms. A homomorphism $u : M \to X$ of right $R-$modules is an $\mathcal{X}-$envelope of a module M provided that
(1) $X \in \mathcal{X}$; and, for every homomorphism $u' : M \to X'$ with $X' \in \mathcal{X}$, there exists a homomorphism $f : X \to X'$ such that $u' = fu$;

$$\begin{array}{ccc} M & \xrightarrow{u} & X \\ \downarrow{\scriptstyle u'} & \swarrow{\scriptstyle f} & \\ X' & & \end{array}$$

(2) $u = fu$ implies that f is an automorphism for every endomorphism $f : X \to X$.
If (1) holds, then $u : M \to X$ is called an $\mathcal{X}-$pre-envelope.

In [3], Guil Asensio, Keskin and Srivastava introduced the notion of $\mathcal{X}$-automorphism-invariant modules. It shows that the endomorphism ring of an $\mathcal{X}$-automorphism-invariant module is a semiregular ring. Some other properties of $\mathcal{X}$-automorphism-invariant modules are studied.

A right $R-$module M having an $\mathcal{X}-$envelope $u : M \to X$ is said to be $\mathcal{X}-$endomorphism-invariant ($\mathcal{X}-$automorphism-invariant) if for any endomorphism (resp., automorphism) g of X, there exists an endomorphism f of M such that $u \circ f = g \circ u$.

In this paper, we introduce the notion of $\mathcal{X}$-nilpotent-invariant modules with accompanying conditions and study some of their properties.

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2 Results

It is easy to see that the $\mathcal{X}-$envelope is unique up to isomorphisms. We have the following propositions.

Theorem 2.1. *([4], Proposition 1.2.1). If $u : M \to X$ and $u' : M \to X'$ are two $\mathcal{X}-$envelopes of an $R-$module M, then $X' \cong X$.*

Theorem 2.2. *([4], Theorem 1.2.5). Let $M = M_1 \oplus M_2$, and $u_i : M_i \to X_i$ are $\mathcal{X}-$envelope of M_i. Then, $u_1 \oplus u_2 : M \to X_1 \oplus X_2$ is a $\mathcal{X}-$envelope of M.*

Let M, N be $R-$modules. We will say that M is $\mathcal{X}-N-$injective if there exist $\mathcal{X}-$envelopes $u_N : N \to X_N$, $u_M : M \to X_M$ satisfying that for any homomorphism $g : X_N \to X_M$, there is a homomorphism $f : N \to M$ such that $gu_N = u_M f$:

$$\begin{array}{ccc} X_N & \xrightarrow{g} & X_M \\ {\scriptstyle u_N}\uparrow & & {\scriptstyle u_M}\uparrow \\ N & \xrightarrow{f} & M \end{array}$$

If M is $\mathcal{X} - M-$injective, then M is said to be an $\mathcal{X}-$endomorphism invariant module.

A class $\mathcal{X}$ of right modules over a ring R, closed under isomorphisms is called an enveloping class if any right $R-$module M has an $\mathcal{X}-$envelope.

Lemma 2.3. *Let $\mathcal{X}$ be an enveloping class. If M is $\mathcal{X} - N-$injective, then M' is $\mathcal{X} - N'-$injective for any direct summand N' of N and any direct summand M' of M.*

Proof. Assume that $N = N' \oplus K, M = M' \oplus L$ for some submodules K of N and L of M. Let $u_{N'} : N' \to X_{N'}, u_K : K \to X_K, u_{M'} : M' \to X_{M'}, u_L : L \to X_L$ be $\mathcal{X}-$envelopes of N', K, M', L, respectively. We have $u_{N'} \oplus u_K : N \to X_{N'} \oplus X_K$ is $\mathcal{X}-$envelope of N and $u_{M'} \oplus u_L : M \to X_{M'} \oplus X_L$ is $\mathcal{X}-$envelope of M. Let $\alpha : X_{N'} \to X_{M'}$ be any homomorphism. Call $\pi : X_{N'} \oplus X_K \to X_{N'}$ the canonical projection and $i : X_{M'} \to X_{M'} \oplus X_L$ the inclusion map. Let $g = i\alpha\pi : X_{N'} \oplus X_K \to X_{M'} \oplus X_L$. Since N is $\mathcal{X} - M-$injective, there exists a homomorphism $f : N \to M$ such that $g(u_{N'} \oplus u_K) = (u_{M'} \oplus u_L)f$.

$$\begin{array}{ccc} X_N & \xrightarrow{g} & X_M \\ {\scriptstyle u_{N'} \oplus u_K}\uparrow & & {\scriptstyle u_{M'} \oplus u_L}\uparrow \\ N & \xrightarrow{f} & M \end{array}$$

It follows that $gu_{N'} = u_{M'}f$. Call $\pi_{M'} : M' \oplus L \to M'$ the canonical projection and $i_{N'} : N' \to N' \oplus K$ the inclusion map. Let $g' = \pi_{M'} f i_{N'} : N' \to M'$. Then we can check that $\alpha u_{N'} = u_{M'} g'$.

$$\begin{array}{ccc} X_{N'} & \xrightarrow{\alpha} & X_{M'} \\ {\scriptstyle u_{N'}}\uparrow & & {\scriptstyle u_{M'}}\uparrow \\ N' & \xrightarrow{g'} & M' \end{array}$$

Therefore, M' is $\mathcal{X} - N'-$injective □

The following corollaries are straightforward and we state them without any proof.

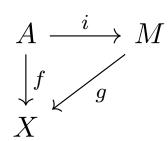
Corollary 2.4. *If M is a $\mathcal{X} - N-$injective module and L is a direct summand of N, then M is an $\mathcal{X} - L-$injective module.*

Corollary 2.5. *Every direct summand of an $\mathcal{X} - M-$injective module is also an $\mathcal{X} - M-$injective module.*

Corollary 2.6. *Any direct summand of an $\mathcal{X}-$ endomorphism invariant module is $\mathcal{X}-$endomorphism invariant.*

Corollary 2.7. *Assume that $M = M_1 \oplus M_2$. If M is $\mathcal{X}-$endomorphism invariant, then M_1 is $\mathcal{X} - M_2-$injective and M_2 is $\mathcal{X} - M_1-$injective.*

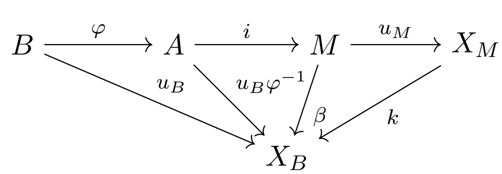
Definition 2.8. An $R-$module M is called *strongly $\mathcal{X}$-pure* if every submodule A of M and any homomorphism $f : A \to X$, with $X \in \mathcal{X}$, extends to a homomorphism $g : M \to X$ such that $gi = f$ in which $i : A \to M$ is the inclusion map



A module M is called a *$C2$-module* if every submodule A of M such that A is isomorphic to a direct summand of M, then A is a direct summand of M.

Theorem 2.9. *Let $\mathcal{X}$ be an enveloping class and let M be a strongly $\mathcal{X}$-pure module. If M is an $\mathcal{X}-$endomorphism invariant module then M is a $C2$-module.*

Proof. Let A be a submodule of M, and B be a direct summand of M such that $A \cong B$. Take $\varphi : B \to A$ an isomorphism. Let $u_B : B \to X_B$ be an $\mathcal{X}-$envelope of B, $u_M : M \to X_M$ be an $\mathcal{X}-$envelope of M. Since M is a strongly $\mathcal{X}$-pure module, the homomorphism $u_B\varphi^{-1} : A \to X_B$ extends to a homomorphism $\beta : M \to X_B$ such that $u_B\varphi^{-1} = \beta i$. Since $u_M : M \to X_M$ is an $\mathcal{X}-$preenvelope of M, there exists $k : X_M \to X_B$ such that $\beta = ku_M$.



We have that M is an $\mathcal{X}-$endomorphism invariant module and B is a direct summand of M and obtain that B is $\mathcal{X} - M-$ injective by Corollary 2.5. Therefore, there exists $f : M \to B$ such that $ku_M = u_B f$

$$\begin{array}{ccc} X_M & \xrightarrow{k} & X_B \\ {\scriptstyle u_M}\uparrow & & {\scriptstyle u_B}\uparrow \\ M & \xrightarrow{f} & B \end{array}$$

Now we have

$$u_B\varphi^{-1} = \beta i = ku_M i = u_B f i$$

As u_B is monomorphism, and so $\varphi^{-1} = fi$. It follows that $i\varphi$ is a splitting monomorphism. That means $Im(i\varphi) = A$ is a direct summand of M. □

Definition 2.10. A right $R-$module M having a $\mathcal{X}-$envelope $u : M \to X$ is said to be *$\mathcal{X}-$nilpotent invariant* if for any nilpotent endomorphism g of X, there exists an endomorphism f of M such that $uf = gu$.

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ {\scriptstyle u}\uparrow & & {\scriptstyle u}\uparrow \\ M & \xrightarrow{f} & M \end{array}$$

Lemma 2.11. *Let $M = M_1 \oplus M_2$ be an $\mathcal{X}-$nilpotent invariant module. Then M_1 is $\mathcal{X} - M_2-$injective.*

Proof. Let $u_1 : M_1 \to X_1$, $u_2 : M_2 \to X_2$ be $\mathcal{X}-$envelopes of M_1, M_2, respectively. Thus, $u = u_1 \oplus u_2 : M \to X = X_1 \oplus X_2$ is an $\mathcal{X}-$envelope of M. For any homomorphism $g : X_1 \to X_2$, $\bar{g} : X \to X$ via $\bar{g}(x_1 + x_2) = g(x_1)$ is a nilpotent endomorphism of X. Since X is an $\mathcal{X}-$nilpotent invariant module, there exists $h : M \to M$ such that $\bar{g}u = uh$. Take $f = \pi_2 h i_1$ with $\pi_2 : M \to M_2$ the canonical projection and $i_1 : M_1 \to M$ the inclusion map, then we have $u_2 f = g u_1$

$$\begin{array}{ccc} X_1 & \xrightarrow{g} & X_2 \\ {\scriptstyle u_1}\uparrow & & {\scriptstyle u_2}\uparrow \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

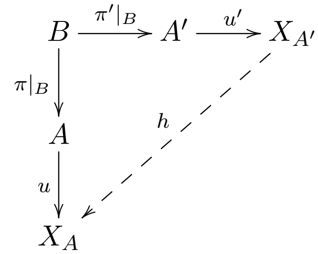
Therefore, M_2 is $\mathcal{X} - M_1-$ injective. □

A module M is called a *C*3*-module* if for two direct summands A and B of M with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M.

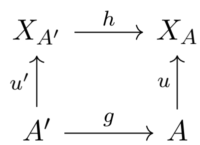
Theorem 2.12. *Let $\mathcal{X}$ be an enveloping class and M be an $\mathcal{X}-$nilpotent invariant module. If M is a strongly $\mathcal{X}$-pure module, then M is a $C3$-module.*

Proof. Assume that A, B are direct summands of M with $A \cap B = 0$. Let A' be some submodule of M such that $M = A \oplus A'$. We claim that, there exists $M' \leq M$ such that $M = A \oplus M'$ and $B \leq M'$. Let $\pi : M \to A$, $\pi' : M \to A'$ be the projections. Since $A \cap B = 0$, $\pi'|_B : B \to A'$ is a monomorphism. Moreover, M is a strongly $\mathcal{X}$-pure module, A' is too.

It follows that $u\pi'|_B : B \to X_{A'}$ is preenvelope, where $u : A \to X_A$ and $u' : A' \to X_{A'}$ are envelopes.



By definition of preenvelope, there exists $h : X_{A'} \to X_A$ such that $hu'\pi'|_B = u\pi|_B$.



Since A is $\mathcal{X} - A'$−injective by Lemma 2.11, there exists $g : A' \to A$ such that $hu' = ug$. Therefore $u\pi|_B = hu'\pi'|_B = ug\pi'|_B$. As u is a momomorphism, $\pi|_B = g\pi'|_B$.
Let

$$M' = \{a' + g(a') \,|\, a' \in A'\}.$$

For every $b \in B$, we have

$$b = \pi'(b) + \pi(b) = \pi'(b) + g\pi'(b).$$

It follows that $b \in M'$. Then $B \leq M'$. It is easy to see that $A \cap M' = 0$ and for every $m \in M$

$$m = a + a' = a - g(a') + (a' + g(a')) \in A + M, (a \in A, a' \in A').$$

Thus $M = A + M'$, and so $M = A \oplus M'$. On the other hand, we have $M = B \oplus B'$ for some $B' \leq M$, then $M' = B \oplus (M' \cap B')$. We deduce that

$$M = A \oplus M' = A \oplus B \oplus (M' \cap B')$$

It means that $A \oplus B$ is a direct summand of M. □

We will say that M is $\mathcal{X}-$*extending invariant (or X-extending)* if there exists an $\mathcal{X}-$envelope $u : M \to X$ such that for any idempotent $g \in End(X)$ there exists an idempotent $f : M \to M$ such that $g(X) \cap u(M) = uf(M)$ or $uf = guf$.

A ring R is called *nil-clean* with any $x \in R$, we can write the form $x = e + n$ for some idempotent $e \in R$ and nilpotent $n \in R$.

Theorem 2.13. *Let M be an $\mathcal{X}-$extending invariant module and $u : M \to X$ be a monomorphic $\mathcal{X}-$envelope with $u(M)$ essential in X.*
Assume that $End(X)$ is a nil-clean ring. If M is $\mathcal{X}-$automorphism invariant and is strongly $\mathcal{X}$-pure then M is an $\mathcal{X}-$endomorphism invariant module.

Proof. Let g be any endomorphism of X. By the hypothesis, we have $g = e + f$ in which e is an idempotent endomorphism of X and f is a nilpotent endomorphism of X. Since M be an $\mathcal{X}-$nilpotent invariant module, there exists a homomorphism $\alpha : M \to M$ such that $fu = u\alpha$.

On the other hand, since M is an $\mathcal{X}-$extending invariant module and e is an idempotent endomorphism of X, there exists an idempotent endomorphism e' of X such that $e(X) \cap u(M) = ue'(M)$. Therefore $A = u^{-1}(e(X)) \cap M = e'(M)$ is a direct summand of M. Since $(1 - e)$ is also an idempotent endomorphism of X, $B = u^{-1}((1 - e)(X)) \cap M$ is a direct summand of M. It is easy to see that $A \cap B = 0$. By Theorem 2.12, $A \oplus B$ is a direct summand of M. Let $M = A \oplus B \oplus C$ for some $C \leq M$, then $M = A \oplus A'$ where $A' = B \oplus C \geq B$. Let $\pi : A \oplus A' \to A$ be the canonical projection. We show that $eu = u\pi$. Assume that, there exists $0 \neq m \in M$ such that $(eu - u\pi)(m) \neq 0$. Since $u(M) \leq^e X$, there exists $m_1 \in M$ such that $u(m_1) = (eu - u\pi)(m) \neq 0$. Hence $u(m_1 + \pi(m)) = eu(m) \in e(X)$, so $m_1 + \pi(m) \in A$. Moreover, $eu(m_1 + \pi(m)) = e^2u(m) = eu(m)$, so $eu(m_1 + \pi(m) - m) = 0$. Now,

$$u(m_1 + \pi(m) - m) = (1 - e)u(m_1 + \pi(m) - m) \in (1 - e)(X),$$

so $m_1 + \pi(m) - m \in B$.

Let $m = a + a'$, where $a \in A, a' \in A'$, then $m_1 + \pi(m) - a - a' \in B \leq A'$ and $\pi(m) = a$. Therefore $m_1 + \pi(m) - a \in A' \cap A = 0$. Thus $m_1 = 0$, a contradiction.

Let $h = \alpha + \pi \in End(M)$, it follows that

$$gu = (e + f)u = eu + fu = u\pi + u\alpha = u(\pi + \alpha) = uh.$$

That means M is an $\mathcal{X}-$endomorphism invariant module.

□

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TÓM TẮT

LỚP CÁC MÔĐUN BẤT BIẾN LŨY LINH

Đinh Đức Tài
Phòng Khoa học và Hợp tác quốc tế, Trường Đại học Vinh
Ngày nhận bài 18/10/2021, ngày nhận đăng 13/12/2021

Một môđun được gọi là bất biến lũy linh nếu nó bất biến dưới mọi tự đồng cấu lũy linh của bao nội xạ chính nó. Các tính chất của lớp môđun này đã được nhiều tác giả nghiên cứu như M. T. Ko¸san và T. C. Quynh trong [8]. Trong bài báo này, chúng tôi giới thiệu một số kết quả khác về lớp môđun này.

Từ khóa: Tự cấu xạ bất biến; bất biến tự đồng cấu; bất biến lũy linh.