A NOTE ON CS-SEMISIMPLE RING

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ABSTRACT

In this paper, we introduce condition (∗) for modules and show that every CS-semisimple ring having condition (∗) is a QF ring.

Keywords: CS-module; uniform module; CS-semisimple ring; Artinian ring; QF ring.

1. Introduction

In this paper, all rings are associative with identity, and all modules are unital right modules. By M_R ($_RM$) we indicate that M is a right (left) module over a ring R. The endmorphism ring of M and the uniform dimension are denoted by $End(M)$ and $u\text{-}dim(M)$, respectively. If the composition length of a module M is finite, then we denoted its length by $l(M)$. For a module M (over a ring R), we consider the following conditions:

(C_1): Every submodule of M is essential in a direct summand of M.

(C_2): Every submodule isomorphic to a direct summand of M is itself a direct summand of M.

(C_3): For any direct summands A, B of M with $A \cap B = 0$, $A \oplus B$ is also a direct summand of M.

A module M is called a *CS-module* if it satisfies the ($C1$) condition. If M satisfies ($C1$) and ($C3$), then M is said to be a *quasi-continuous module*. M is defined to be a *continuous module* if it satisfies the ($C1$) and ($C2$) conditions. We have the following implications:

Injective ⇒ quasi-injective ⇒ continuous ⇒ quasi-continuous ⇒ CS

We refer to [2], and [4] for background on CS and (quasi-)continuous modules.

A ring R is called a *right (or left) continuous (CS, quasi-continuous)* if R_R (resp. $_RR$) is continuous module (resp. CS, quasi-continuous). A ring R is defined to be *CS-semisimple* if every right (or left) R-module is CS. A ring R is called *QF* if R is a right and left self-injective, right and left Artinian ring.

We refer to [2] for background on CS-semisimple and QF ring.

It is well-known (see, e.g., [2, 18.1]) that a ring R is QF iff R is right or left self-injective, right or left Artinian; iff R is right or left self-injective, right or left Noetherian. In [2, 13.5], if R is a CS-semisimple ring then R is right and left Artinian. A natural question, if R is a CS-semisimple, is R necessarily QF? In [6], [7], there exists a CS - semisimple ring, but is not QF. In this paper, we are interested in the following question: "*When CS-semisimple ring is QF?*". The question will be partially answered in Theorem 3.1.

Now, we consider the class Abel groups (ie., the class $\mathbb{Z}-$modules where $\mathbb{Z}$ is the ring of integer numbers). Let $M = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$ be a direct summand of two uniform module $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$ with the finite composition length, then $\mathbb{Z}/2\mathbb{Z}$ embeded in $\mathbb{Z}/4\mathbb{Z}$. But $\mathbb{Z}-$module, $N = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z})$ be also a direct summand of two uniform module $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/9\mathbb{Z}$ with the finite composition length, then $\mathbb{Z}/2\mathbb{Z}$ does not embed in $\mathbb{Z}/9\mathbb{Z}$. In general, let R be a ring and $M = \oplus_{i\in I} M_i$ be a direct sum of uniform right $R-$submodule M_i with the composition length of the module M_i are finite for all $i \in I$. We consider the following condition for a given M: "$(^*)$ *For all $i, j \in I$ and $i \neq j$ such that $l(M_i) < l(M_j)$ then M_i does not embed in M_j*". In section 2, we give some examples the module M satisfy the $(^*)$ condition. We show that if R is a right continuous and right Artinian ring, then R_R satisfies the $(^*)$ condition (see Proposition 2.2).

In section 3, we show that every CS-semisimple ring having condition $(^*)$ is a QF ring (see Theorem 3.1), and give examples or either CS-semisimple rings or rings having condition $(^*)$, but is not QF.

2 The $(^*)$ condition

In this section, we introduce condition $(^*)$ for modules and give some results about modules and rings having condition $(^*)$.

Let $M = \oplus_{i\in I} M_i$ be a direct sum of uniform submodule M_i with the composition length of a module M_i are finite for all $i \in I$. We consider the following condition for a given M:

$(^*)$ *For all $i, j \in I$ and $i \neq j$ such that $l(M_i) < l(M_j)$ then M_i does not embed in M_j.*

Example 2.1. *(i) Let*

$$M = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z}) \oplus (\mathbb{Z}/125\mathbb{Z}) \oplus ... \oplus (\mathbb{Z}/p_n^n\mathbb{Z}) \oplus ...$$

with $2 = p_1, 3 = p_2, 5 = p_3, ..., p_n, ...$ are prime numbers; then M is right and left $\mathbb{Z}-$module satisfies the $(^*)$ condition.

(ii) Let $N = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ then N does not satify the $(^*)$ condition.

Let R be a ring such that $R_R = e_1R \oplus ... \oplus e_nR$ where each eR is a uniform right ideal with $l(e_iR) < \infty$ and $\{e_i\}_{i=1}^n$ is a system of idempotents. If R_R satisfies the $(^*)$ condition, we called R is *right $(^*)$ ring*. Similarly we can define *left $(^*)$ ring*. The example, ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_9$ is left and right $(^*)$ ring but $R = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ is not right (and left) $(^*)$ ring.

Let R be a right Artinian right CS ring, then R is also right Noetherian. By [8] (or [4, Theorem 2.19]), $R_R = R_1 \oplus ... \oplus R_2$ with R_i is uniform right ideal of R for all $i = 1, ..., n$. Note that $l(R_i) < \infty$ for all $i = 1, ..., n$. When R is right $(^*)$ ring?

Proposition 2.2. *Let R be a right continuous and right Artinian ring, then R is right $(^*)$ ring.*

Proof. Let R be a right continuous, right Artinian ring, then since R is right CS and right Noetherian , we must have

$$R = R_1 \oplus ... \oplus R_n$$

where R_i is a right uniform ideal of R for all $i = 1, 2, ..., n$ (by [8] or [4, Theorem 2.19]). Note that, R_i is a direct summand of R_R, it follow that R_i is a Artinian and Noetherian right ideal of R, thus $l(R_i) < \infty$ for all $i = 1, 2, ..., n$ (by [5, 32.4 (1)]). We show that R_i does not embed in R_j for all $i, j = 1, ..., n$ and $i \neq j$ such that $l(R_i) < l(R_j)$. Set $M = R_i \oplus R_j$, thus M is continuous right $R-$module by [4, Proposition 2.7]. Let $f : R_i \longrightarrow R_j$ be a monomorphism with $f(R_i) = P$ is a submodule of R_j; then $P \cong R_i$. Note that $l(P) = l(R_i) < l(R_j)$, thus P is a proper submodule of R_j. Since, R_i is a direct summand of M and M has (C_2) property, $M = P \oplus Q$. By modularity we get $R_j = R_j \cap M = R_j \cap (P \oplus Q) = P \oplus T$ where $T = Q \cap R_j$. Since P is a proper submodule R_j, thus T is a nonzero submodule of R_j, a contradiction. Therefore R_i does not embed in R_j, as desired. □

The opposite direction of Proposition 2.2 is unknown. We have the following open question:

Question 2.3. *Let R be a right CS, right $(^*)$ and right Artinian ring. Is R necessarly right continuous ring?*

3 When CS-semisimple ring is QF

In this section, based on Section 2, we show that a ring R is QF if R is CS-semisimple and right $(^*)$ ring (Theorem 3.1).

Theorem 3.1. *Let R be a CS - semisimple ring and right $(^*)$ ring then R is a QF ring.*

Proof. Let R be a CS - semisimple ring and right $(^*)$ ring, we show that R is QF. Theo [2, 13.5], we have R is right and left Artinian and

$$R = R_1 \oplus R_2 \oplus ... \oplus R_n,$$

with R_i are minimal right ideals or uniform injective right ideals of composition length 2. Assume that, $R_1, ..., R_k$ (with $k \leq n$) are minimal right ideals and $R_{k+1}, ..., R_n$ are uniform injective right ideals of composition length 2. Set $S = R_1 \oplus ... \oplus R_k$ and $T = R_{k+1} \oplus ... \oplus R_n$. Clearly S is semisimple Artinian and T has a zero right socle.

Step 1. If $T = 0$ then $R = S = Soc(R_R)$, R is a semisimple Artinian ring. This implies that R is a QF ring.

Step 2. If $S = 0$, then $l(R_1) = ... = l(R_n) = 2$. We have shown that:

(i). For $1 \leq i < j \leq n$*,* $R_i \oplus R_j$ *are quasi - continuous module.*

By [5, 32.4 (3, (iii))], $End(R_i)$ and $End(R_j)$ are local rings. We show that $X = R_i \oplus R_j$ satisfies the condition (C_3). Let X_1 and X_2 nonzero two direct summands of X with $X_1 \cap X_2 = 0$. We show that $X_1 \oplus X_2$ is also a direct summand of X. Indeed, since $u - dim(X) = 2$, we can assume that both X_1 and X_2 are uniform. Write $X = X_2 \oplus C$. By Azumaya's generalization of the Krull-Schmidt Theorem (cf. [1, 12.6, 12.7]), either $X = X_2 \oplus C = X_2 \oplus R_i$, or $X = X_2 \oplus C = X_2 \oplus R_j$. Since i and j can interchange with each other, we need only consider one of the two possibilities. Let us consider the case $X = X_2 \oplus C = X_2 \oplus R_i = R_i \oplus R_j$. Then it follows

$$X_2 \cong R_j.$$

Write, $X = X_1 \oplus D$, thus $X = X_1 \oplus D = X_1 \oplus R_i$, or $X = X_1 \oplus D = X_1 \oplus R_j$.

If $X = X_1 \oplus R_i$, then by modularity we get $X_1 \oplus X_2 = (X_1 \oplus X_2) \cap X = (X_1 \oplus X_2) \cap (X_1 \oplus R_i) = X_1 \oplus E$ where $E = (X_1 \oplus X_2) \cap R_i$ is a submodule of R_i. From here we get $E \cong X_2 \cong R_j$, this means R_i contains a copy of $X_2 \cong R_j$. Since $2 = l(R_j) = l(X_2) = l(E) = l(R_i)$, we must have $E = R_i$, and hence $X_1 \oplus X_2 = X_1 \oplus R_i = X$.

If $X = X_1 \oplus R_j$, then by modularity we get $X_1 \oplus X_2 = (X_1 \oplus X_2) \cap X = (X_1 \oplus X_2) \cap (X_1 \oplus R_j) = X_1 \oplus F$ where $F = (X_1 \oplus X_2) \cap R_j$ is a submodule of R_j. From here we get $F \cong X_2 \cong R_j$, this means R_j contains a copy of $X_2 \cong R_j$. Since $2 = l(R_j) = l(X_2) = l(F) = l(R_j)$, we must have $F = R_j$, and hence $X_1 \oplus X_2 = X_1 \oplus R_j = X$.

Therefore, in any case, we obtain that $X_1 \oplus X_2$ is a direct summand of X, i.e., X satisfies the condition (C_3). By hypothesis, R is a CS - semisimple ring, thus X is a CS module, it follows that M is a quasi-continuous module.

(ii) R *is a right self-injective.*

By hypothesis of R_i, R_i is R_i-injective for all $i = 1, 2, ..., n$. By (i), $R_i \oplus R_j$ quasi-continuous right $R-$module, thus R_i is R_j-injective for all $i, j = 1, ..., n$ and $i \neq j$ (see [4, Proposition 2.10]). Since [4, Corollary 1.19], R is a right self-injective, as required. By hypothesis, R is right Artinian, it follow that, R is a QF ring.

Step 3. If $S \neq 0$ and $T \neq 0$, ie., $1 \leq k < n$. We have shown that:

(iii). For $1 \leq i < j \leq n$*,* $R_i \oplus R_j$ *are quasi - continuous module.*

If $1 \leq i < j \leq k$ is clear and $k + 1 \leq i < j \leq n$ follow that by (i). Now, assume that $1 \leq i \leq k < j \leq n$, set $Y = R_i \oplus R_j$. Then Y is a direct summand of R_R. By hypothesis R is a CS-semisimple and right $(*)$ ring, thus Y is a CS-module and R_i does not embed in R_j. Let Y_1 and Y_2 nonzero two direct summands of Y with $Y_1 \cap Y_2 = 0$. We show that $Y_1 \oplus Y_2$ is also a direct summand of Y. Indeed, since $u - dim(Y) = 2$, we can assume that both Y_1 and Y_2 are uniform. Note that, $End(R_i)$ and $End(R_j)$ are local rings (see [5, 32.4 (3, (iii))]). Write $Y = Y_2 \oplus P$. By Azumaya's generalization of the Krull-Schmidt Theorem (cf. [1, 12.6, 12.7]), either $Y = Y_2 \oplus P = Y_2 \oplus R_i$, or $Y = Y_2 \oplus P = Y_2 \oplus R_j$.

Case 1. If $Y = Y_2 \oplus P = Y_2 \oplus R_i = R_i \oplus R_j$. Then it follows

$$Y_2 \cong R_j.$$

Write, $Y = Y_1 \oplus Q$, thus $Y = Y_1 \oplus Q = Y_1 \oplus R_i$, or $Y = Y_1 \oplus Q = Y_1 \oplus R_j$.

If $Y = Y_1 \oplus R_i$, then by modularity we get $Y_1 \oplus Y_2 = (Y_1 \oplus Y_2) \cap Y = (Y_1 \oplus Y_2) \cap (Y_1 \oplus R_i) = Y_1 \oplus K$ where $K = (Y_1 \oplus Y_2) \cap R_i$ is a submodule of R_i. From here we get $K \cong Y_2 \cong R_j$, this means R_i contains a copy of $X_2 \cong R_j$. Since $2 = l(R_j) = l(Y_2) = l(K) \leq l(R_i) = 1$, this is a contradiction.

If $Y = Y_1 \oplus R_j$, then by modularity we get $Y_1 \oplus Y_2 = (Y_1 \oplus Y_2) \cap Y = (Y_1 \oplus Y_2) \cap (Y_1 \oplus R_j) = Y_1 \oplus L$ where $L = (Y_1 \oplus Y_2) \cap R_j$ is a submodule of R_j. From here we get $L \cong X_2 \cong R_j$, this means R_j contains a copy of $Y_2 \cong R_j$. Since $2 = l(R_j) = l(X_2) = l(L) = l(R_j)$, we must have $L = R_j$, and hence $Y_1 \oplus Y_2 = Y_1 \oplus R_j = Y$.

Case 2. If $Y = Y_2 \oplus P = Y_2 \oplus R_j = R_i \oplus R_j$. Then it follows

$$Y_2 \cong R_i,$$

i.e., Y_2 is a simple right $R-$module. Write, $Y = Y_1 \oplus T$, thus $Y = Y_1 \oplus T = Y_1 \oplus R_i$, or $Y = Y_1 \oplus T = Y_1 \oplus R_j$.

If $Y = Y_1 \oplus R_j$, then by modularity we get $Y_1 \oplus Y_2 = (Y_1 \oplus Y_2) \cap Y = (Y_1 \oplus Y_2) \cap (Y_1 \oplus R_j) = Y_1 \oplus G$ where $G = (Y_1 \oplus Y_2) \cap R_j$ is a submodule of R_j. From here we get $G \cong Y_2 \cong R_i$, this means R_j contains a copy of $Y_2 \cong R_i$. Thus R_i embeded in proper submodule of R_j, this is a contradiction because Y satisfies the $(^*)$ condition.

If $Y = Y_1 \oplus R_i$, then by modularity we get $Y_1 \oplus Y_2 = (Y_1 \oplus Y_2) \cap Y = (Y_1 \oplus Y_2) \cap (Y_1 \oplus R_i) = Y_1 \oplus H$ where $H = (Y_1 \oplus Y_2) \cap R_i$ is a submodule of R_i. From here we get $H \cong Y_2 \cong R_i$, this means H is a simple submodule of R_i. Since R_i is also simple module, we must have $H = R_i$, and hence $Y_1 \oplus Y_2 = Y_1 \oplus R_i = Y$.

Therefore, in any case, we obtain that $Y_1 \oplus Y_2$ is a direct summand of Y, i.e., Y satisfies the condition (C_3), but Y is a CS - module, thus Y is quasi - continuous module, finishing the proof (iii).

(iv) R is a right self-injective.

By hypothesis of R_i, R_i is R_i-injective for all $i = k+1, ..., n$ and with $i = 1, ..., k$ the following cases are trivial. By (iii), $R_i \oplus R_j$ quasi-continuous right $R-$module, thus R_i is R_j- injective for all $i, j = 1, ..., n$ and $i \neq j$ (see [4, Proposition 2.10]). By hypothesis, R is right Artinian, it follow that, R is a QF ring, finising the proof theorem. □

It is worth mentioning the following note.

Remark 3.2. *Theorem 3.1 is not true, in general,*

(i) There exists a CS-semisimple but it is not a QF ring.

(a) For example, consider the matrix ring of the form

$$R = \begin{bmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} \end{bmatrix},$$

where $\mathbb{R}$ is the field of real numbers. By [7, Remark 3.4], R it is a right CS-semisimple (and hence it is right and left Artinian), but is not right self-injective (since $E(R_R) = \begin{bmatrix} \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \end{bmatrix}$). Therefore, R is not a QF ring.

(b) We consider the matrix ring of the form

$$R = \begin{bmatrix} \mathbb{C} & V & 0 \\ 0 & \mathbb{C} & V \\ 0 & 0 & \mathbb{C} \end{bmatrix},$$

where $\mathbb{C}$ is the field of complex numbers and V is a $\mathbb{C}-$bialgebra with $\dim({}_{\mathbb{C}}V) = \dim(V_{\mathbb{C}}) = 1$ and $V^2 = 0$. By [6, Example 3.9], R is a CS -semisimple ring but it is not QF.

(ii) By [9]or [2, 18.26], a ring R is QF iff R is right and left continuous, right and left Artinian. Since Proposition 2.2, if R is a QF ring then R is right and left $(^*)$ ring. There exist a right $(^*)$ ring but it is not a QF ring. Let F be a field, and let $f : F \longrightarrow K$ be a ring epimorphism, where K is a subfield of F (thus, F is a vector space over the field K) such that $dim(F_K) = \infty$. For example, let $F = \mathbb{Q}(x_1, x_2, ...)$ and $K = \mathbb{Q}(x_1^2, x_2^2, ...)$ with

$$\mathbb{Q}(x_1, x_2, ...) = \{\frac{P(x_1, x_2, ...)}{Q(x_1, x_2, ...)} \mid P(x_1, x_2, ...), Q(x_1, x_2, ...) \in \mathbb{Q}[x_1, x_2, ...]\}$$

and

$$\mathbb{Q}(x_1^2, x_2^2, ...) = \{\frac{P(x_1^2, x_2^2, ...)}{Q(x_1^2, x_2^2, ...)} \mid P(x_1^2, x_2^2, ...), Q(x_1^2, x_2^2, ...) \in \mathbb{Q}[x_1, x_2, ..., x_n...]\},$$

where $\mathbb{Q}$ is the field of rational numbers and $\mathbb{Q}(x_1, x_2, ...)$ is the polynomal ring infinite many indeterminates over the field $\mathbb{Q}$; and let $f(x_i) = x_i^2, \forall i = 1, 2, ..., n, ...,$ $f(a) = a, \forall a \in \mathbb{Q}$. Then F is an $(F, K)-$bimodule, and we consider the matrix ring of the form

$$R = \begin{bmatrix} F & F \\ 0 & K \end{bmatrix},$$

By [3, Example 7.11'.1] and [2, Example 18.27], R is a right continuous and right Artinian and hence R is right $(^*)$ ring, but it is not QF.

4 Conclusion

Let R be a ring and $M = \oplus_{i \in I} M_i$ be a direct sum of uniform right $R-$submodule M_i with the composition length of the module M_i are finite for all $i \in I$. We consider the following condition for a given M: "*For all $i, j \in I$ and $i \neq j$ such that $l(M_i) < l(M_j)$ then M_i does not embed in M_j*". As we know, if R is a CS-semisimple ring then it is not necessarily a QF ring. We are interested in questions: "When CS-semisimple ring is QF?" and we show that if R is a CS-semisimple ring and right $(^*)$ ring then R is a QF ring (Theorem 3.1).

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TÓM TẮT

MỘT CHÚ Ý TRÊN VÀNH CS-NỬA ĐƠN

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Trong bài báo này, chúng tôi giới thiệu điều kiện $(^*)$ cho môđun và chứng minh rằng mọi vành CS-nửa đơn có điều kiện $(^*)$ là một vành QF.
Từ khóa: CS-môđun; môđun đều; vành CS-nửa đơn; vành Artin; vành QF.